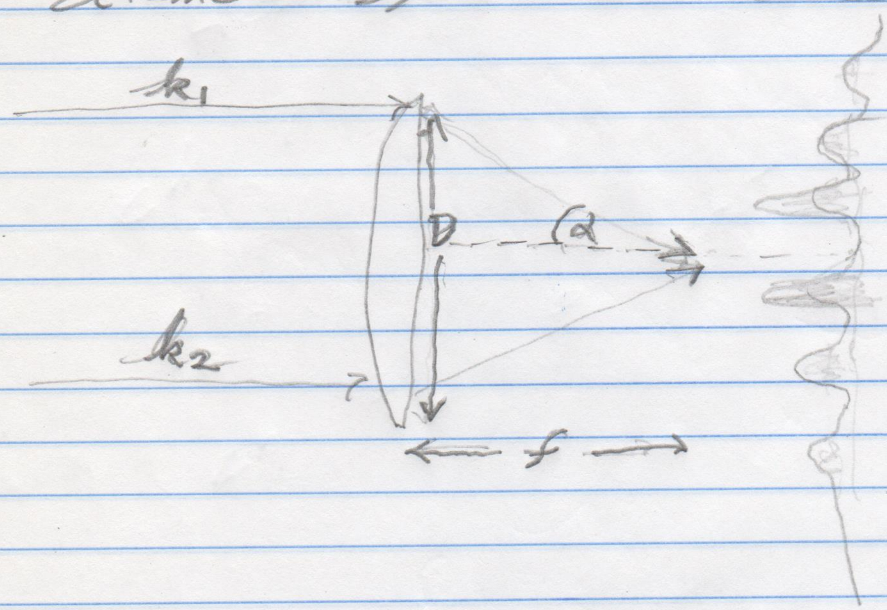
# Ch. 2 Elements of Fourier or Physical optics

1) ~~Motivation:~~

Carl Zeiss comissioned Abbe to make a better microscope. Abbe failed as he tried a a small aperture system. He reported that diffraction was the issue. So he went to a larger diameter system and succeded.




Intensity given by Jinc function

$$\text{Period} = \frac{\lambda}{2\sin\alpha}$$

$$\tan\alpha = \frac{D}{2}/f = \frac{D}{2f}$$

Small angle $\alpha \approx D/2f$

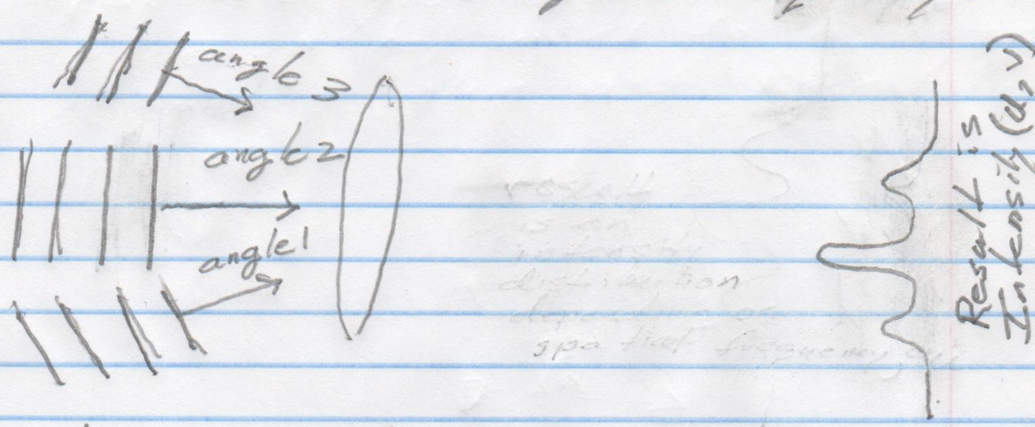
$\sin\alpha \approx D/2f$

Period or spatial frequency

$$= \frac{\lambda}{2 \cdot D/2f} = \frac{\lambda f}{D}$$

Similar to Fraunhofer result

Fourier optics, Light field is written as a superposition of plane waves propogating in different directions. A distribution of angles is akin to "angular frequency.




The wavelets, Haygens principle, are comprised of a superposition of angles. the result of len depends upon the spatial frequency.

Spatial frequency

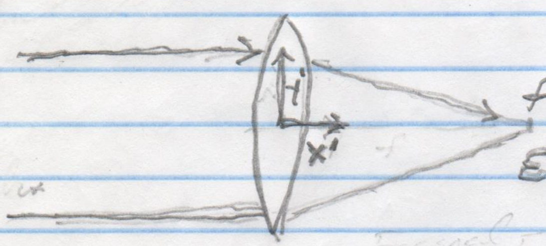
$$u = \frac{x}{f\lambda} \quad , \quad v = \frac{y}{f\lambda}$$

since $f = [\text{length}]$ ; $\lambda = [\text{length}]$

$$u = \left[\frac{1}{\text{length}}\right]$$

# Elements of Fourier or Physical optics. P-3

Fresnel diffraction integral[1]



$$\mathcal{E}^f = \underbrace{\left(\frac{\mathcal{E}_0 e^{ik \cdot r}}{i\lambda f}\right)}_{\text{Fresnel propogator of spence}^2} \mathcal{F}[f(x', y')](u, v)$$

$$r = f + p^2/2f$$

For ease of solution, we split the Fresnel integral into two parts.

1) $f_i(x', y')$ = field incident on lens.
2) $t(x', y')$ = transmision of lens.

With the convolution therom we have:

$$\mathcal{E}^f = \frac{\mathcal{E}_0 e^{ikr}}{i\lambda f} \mathcal{F}[fi[f(x', y')](u v) \,\underbrace{\mathcal{F}[t(x', y')](u v)}_{\text{Point spread function}}$$

$$= \frac{\mathcal{E}_0 e^{ikr}}{i\lambda f} \mathcal{F} fi(x', y') \cdot \text{p.s.f}$$

For a circular lens
point spread, $p(u, v) = \frac{\pi D^2}{4} \, \text{jinc}\left(\frac{\pi D \rho}{\lambda f}\right)$

jinc is Bessel function analogy to sinc $= \frac{\sin x}{x}$


1 Adams + Hughes "optics f2f, from Fresnel to Fourier oxford 2019
2 Spence, High resolution TEM section 3.5

# Elements of Fourier optics 4

For a plane wave on a lens the solution is simple

$$f_i(x', y') = e^{ikx}$$

$$\mathcal{F}[e^{ikx}] = \delta(u) \text{ ; delta function}$$

For two plane waves seperated by an angle, $\Delta\theta$ we have:

$$\mathcal{F}\left[e^{(2\pi i \Delta\theta/\lambda) x'}\right](u) = \delta\left(u - \frac{\Delta\theta}{\lambda}\right)$$

where $\frac{2\pi}{\lambda}(\theta_2 - \theta_1) = k_2 - k_1$ ; $u = \frac{x'}{\lambda f}$ {[illegible]}

Plugging this into Fresnel eq.

$$\mathcal{E}^f = \frac{\mathcal{E}_o e^{ikr}}{i\lambda f} \delta\left(u - \frac{\Delta\theta}{\lambda}\right) \cdot \text{p.s.f.}$$

For a lens of diameter $D$ and focal length $f$;

$$\text{P.S.F.} = \text{jinc}\left(\frac{\pi D \rho}{\lambda f}\right)$$

Intensity $\equiv$ $\mathcal{E}^{f*}\mathcal{E}^f$

$$= I_o \frac{\pi^2 D^4}{16\lambda^2 f^2} \text{jinc}^2 \left\{\pi D \left(u - \frac{\Delta\theta}{\lambda}\right)^2 + v^2\right\}^{1/2}$$

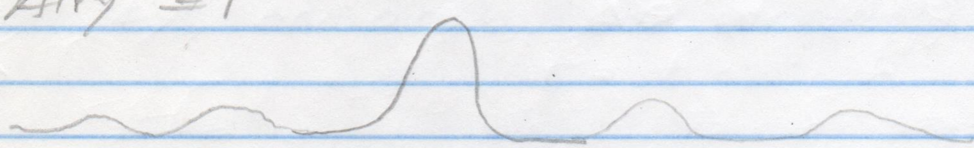
Air Pattern center displaced to $x = \Delta\theta f$

see eq 9.5 Adams + Hughes

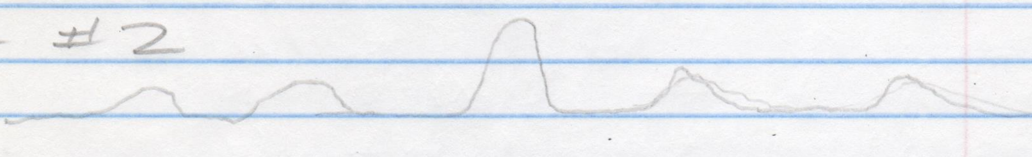
So, the two monochromatic plane waves of slightly different incident angle, $\Delta\theta$, give us a nice result. The observed intensity in the focal plane is two Airy patterns. What is the resolution of the system given these two incident plane waves?

Rayliegh criteria:

Airy #1



Air #2



$$\Rightarrow \quad f\,\Delta\theta = \frac{1.22 f\lambda}{D}$$ (left as homework)

and $$\Delta\theta_R = \frac{1.22\,\lambda}{D}$$

so we get smaller $\Delta\theta$, better resolution when $D$ is increased. That is, when more spatial frequency's are subtended by the lens.

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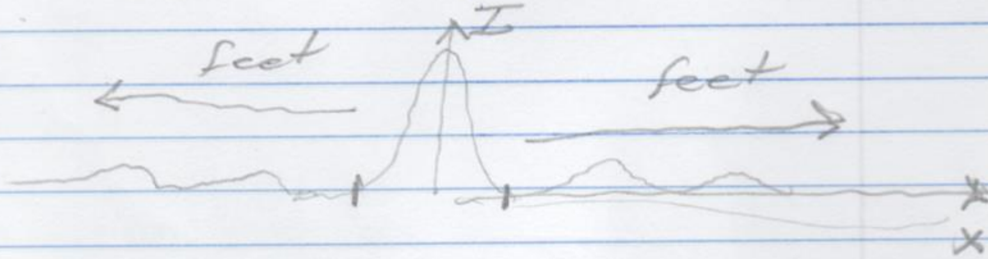
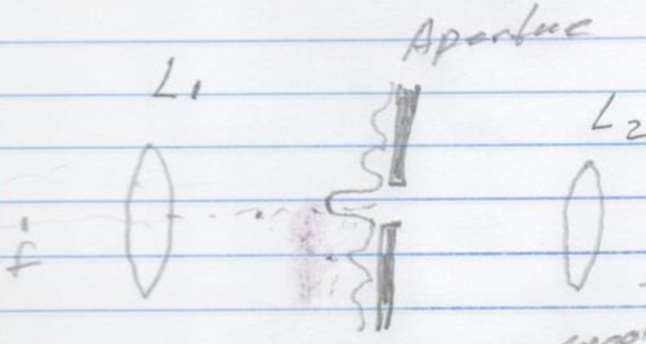
Spatial filtering.

We learned that resolution increases when we increase the diameter of the lens, this confounded Abbe early on.

Can we alter the resolution of a system by changing the spatial frequencies that get to the lens? Yes, concept of apodization, to cut feet off a pod = without feet.




remove Intensities at large x

large x features are high spatial frequency. So, if we remove them with an aperture we smooth the resulting image

smoothed image from apodization

We do this in optical holography, "clean up" the laser beam with a strong lens 1, objective eyepiece, and a pin hole aperture at focal point.

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- We see examples of the Fourier transform of the lens in appendix II, physical optics

Appendix II. fig 1 sinc (sombrero) rectangular aperture

fig 2 Bessel functions note different orders and symmetry

fig 3 Jinc function, $\mathcal{F}T$ of round lens

From the result of fig 4 we see the Jinc and sinc function are very similar. So, if you are familiar with $\text{sinc}(x) = \frac{\sin(x)}{x}$ please use that analogy!

Airy function, The Intensity or E field squared at field pts

$$\mathcal{E}^f = \frac{\mathcal{E}_0 e^{ikz}}{i\lambda f} \mathcal{F}\left[f_i(x',y')\right](u,v) * psf(u,v)$$

9.2 Adams + Hughes

for plane wave $f_i(x',y') = e^{ikx'}$

$$psf(u,v) = \frac{\pi D^2}{4} \text{Jinc}\left(\frac{\pi D \rho}{\lambda f}\right)$$

where $u = \frac{x}{\lambda f}$

next page

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But $\mathcal{F}$ transform of a plane wave = $\delta$ function

$$\therefore \quad \mathcal{E}^{f} = \frac{\mathcal{E}_0 e^{ikz}}{i\lambda f} \, \delta\left(u - \frac{\Delta\theta}{f}\right) * psf(u,v)$$

For a plane wave inclined at an angle $\Delta\theta$ to normal incidence. This example used in Adams + Hughes as an example of angular resolution.

$$\Rightarrow \quad I^{(f)} = \frac{I_0 \pi^2 D^4}{16\lambda^2 f^2} \, \text{Sinc}^2 \, \pi D \left\{ \left(u - \frac{\Delta\theta}{\lambda}\right)^2 + v^2 \right\}^{1/2}$$

See appendix II fig 4, Airy pattern + fig 5 two overlapping Airy patterns.

The angular resolution of a lens is the ~~sep~~ minimum separation of the Air patterns or when max of one pattern is upon min of other

$$\Delta\theta_{\text{Resolution}} = \frac{1.22\lambda}{D}$$

This is what Abbe discovered! We need a bigger lens to improve the microscope.

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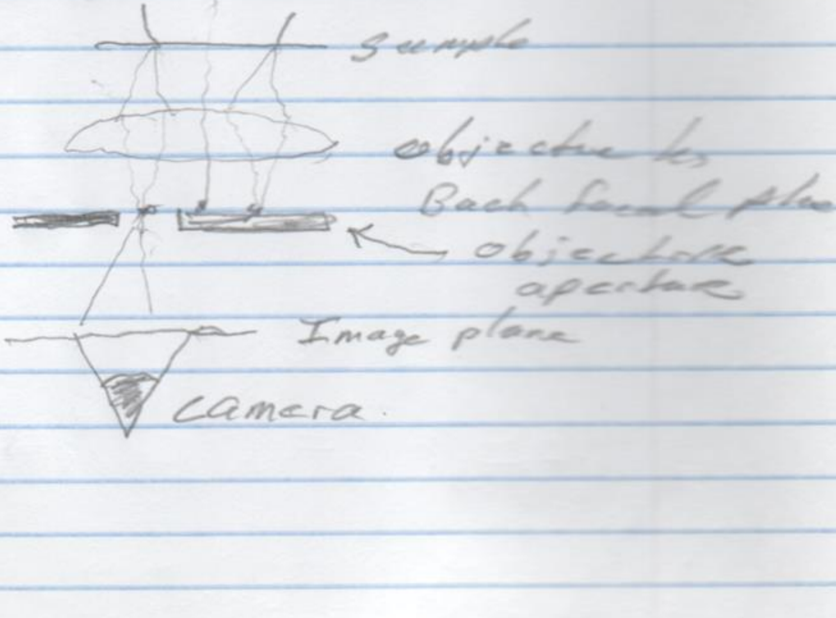
Inverse Apodization, super resolution

You can actually block the central part of a diffraction pattern (image) of lens 1 with an annular filter then re image this with lens 2, to form an image with only the higher spatial frequencies, and "increase" the resolution

Conversly, In dark field TEM we select only a few spatial frequencys. This eliminates the higher spatial frequencys and so the image has lower resolution.



Relationships between Fourier optics concepts + terminology and electron microscopy.

Does the point spread function of page 4, $\mathcal{F}[I(x',y')](u,v)$ which is typically $\frac{\pi D^2}{4} \, jinc\left(\frac{\pi D \rho}{\lambda f}\right)$ Adams Hughes 9.3

equate to contrast transfer function in Electron microscopy of Spence or Carter + Williams

Yes, Spence, has similar approach but the psf, point spread function, of Fourier optics is modulated by a term that represents the strong influence of spherical aberration in electron optics:

Microscope Transfer Function $A(u,v) = \underbrace{P(u,v)}_{\text{pupil transfer function}} \underbrace{\exp j\chi(u)}_{\text{aberration attenuates high spatial frequencies}}$ Spence 3.24a

In Williams + Carter

Intensity $g(x,y)$:

$$g = I(x,y) = \sum_{u} G(u_x, u_y) \exp 2\pi i(x u_x + y u_y)$$

sum over spatial frequency's or for HRTEM $u_x$ are reciprocal lattice vec

contrast transfer function C.T.F. $G(u) = H(u) F(u)$

$= \underbrace{A(u) \, B(u) \, E(u)}_{H} \, F_u$ (28.5)

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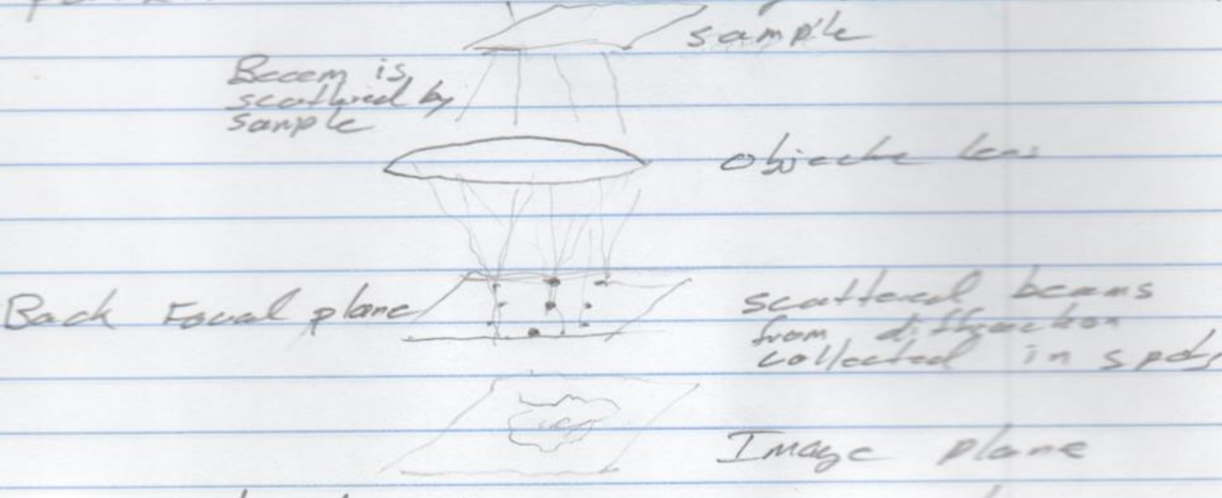
Relationship of ~~sp~~ Fourier optics to modern electron optics terms:

In electron microscope, the electron beam (charged particle) interacts with the charge in the sample. This changes the phase of the ~~wa~~ electron waves. This is a more complicated interaction than the ~~un~~ charged photon (light) passing through a sample and a lens

The connection of Fourier optics to electron microscopy is particularly strong in the TEM. In the back focal plane of the objective lens we have the diffraction pattern or the angular information:

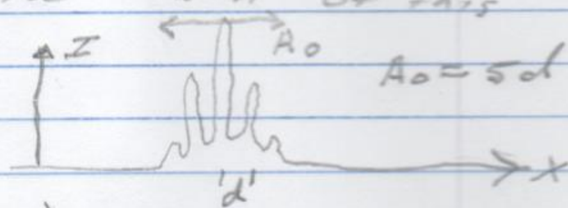



Now, back to Fourier optics

Example, 4f spatial filter.

Or, cleaning up a gaussian laser beam. On the bottom of page 4, I mentioned cleaning up a laser beam with a strong lens + pin hole aperture. Now we see the math of this

Input intensity




$A_0 = 5d$

$$f(x'y') = e^{-(x'^2+y'^2)/A_0^2}\left(1 + \epsilon\cos\left(\frac{2\pi x'}{d}\right)\right)$$

where $A_0$ = Gaussian beam width

$d$ = spacing between the dirty little peaks above

Fresnel Integral:

$$\mathcal{E}^{(f)} = \frac{\mathcal{E}_0 e^{ikf}}{i\lambda f} FT\left\{\left[f(x'y')\right]_{uv}\right\} \qquad 9.1$$

$$FT\left\{f(x',y')_{uv}\right\} = FT\left\{e^{-(x^2+y^2)/A_0^2}\left(1+\epsilon\cos\left(\frac{2\pi x}{d}\right)\right)\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(ux+vy)}\, e^{-(x^2+y^2)/A_0^2}\left\{1+\epsilon\cos\frac{2\pi x}{d}\right\} dx\,dy$$

This is hard to integrate, so lets examine the convolution approach

Cleaning a gaussian beam continued:

Now carry out the convolution:

$$\mathcal{E}^f = \frac{\mathcal{E}_0 e^{ikr}}{i\lambda f}\left\{\delta(u) + \frac{\varepsilon}{2}\delta(u+\tfrac{1}{d}) + \frac{\varepsilon}{2}\delta(u-\tfrac{1}{d})\right\} * \pi A_0^2 e^{-\pi(u^2+v^2)A_0^2}$$

$A_0$ = width of Gaussian beam

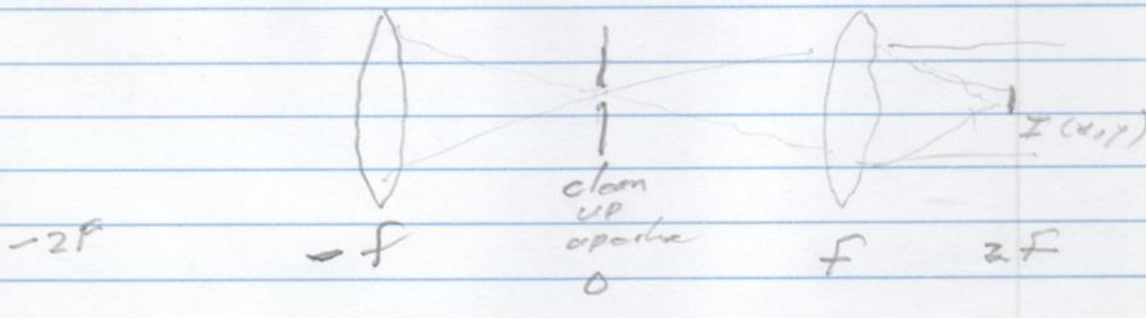
take only central spatial freq,

set $\frac{A_0}{d} = 5 \Rightarrow \frac{1}{d} = 5A_0$

This eliminates the two spots from the displaced $\delta$ functions

$$\mathcal{E}^f = \frac{\varepsilon\,\mathcal{E}_0 e^{ikr}}{2\,i\lambda f}\,\pi A_0^2 e^{-\pi(u^2+v^2)A_0^2}\,\delta(u)$$

Now Fourier Transform again at the first focal plane



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4f spatial filter continued, gaussian beam clean up:

After the second lens, one more F.T.

$$= \text{const} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(ux+vy)} e^{-\pi(u^2+v^2)A_0^2}\, dx\,dy$$

$$\mathcal{E}^f = (\lambda f)^2 \frac{\mathcal{E}_0 e^{ikr}}{i\lambda f} \frac{\mathcal{E}}{2} \pi A_0^2 \int_{-\infty}^{\infty} e^{-2\pi i v y}\, dv \int_{-\infty}^{\infty} e^{-\pi(u^2+v^2)A_0^2} e^{-2\pi i(ux)}\, du$$

compare 9.16 Adams & Hughes

integrale over $dv$ as well? see 4f filter example

from $dx = f\lambda\, du$, $dy = f\lambda\, dv$

Note see 9.6 Adams & Hughes,

$$u = \frac{x}{f\lambda}, \quad v = \frac{y}{f\lambda}$$

Now, use $e^{-i\theta} = \cos\theta - i\sin\theta$

Integral over sine vanishes

Then Abromowitz Stegun p. 302

$$\mathcal{FT}\left[e^{-ax^2}\right](k) = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2 k^2/a}$$

$$a \to \pi A_0^2$$

$$\Rightarrow \mathcal{FT}\left[e^{-\pi u^2 A_0^2}\right] = \sqrt{\frac{\pi}{\pi A_0^2}}\, e^{-\pi^2 x^2/\pi A_0^2}$$

$$= \frac{1}{A_0} e^{-\pi x^2/A_0^2}$$

$A_0$ = width of gaussian, so $x^2/A_0^2$ has no dimensions ✓

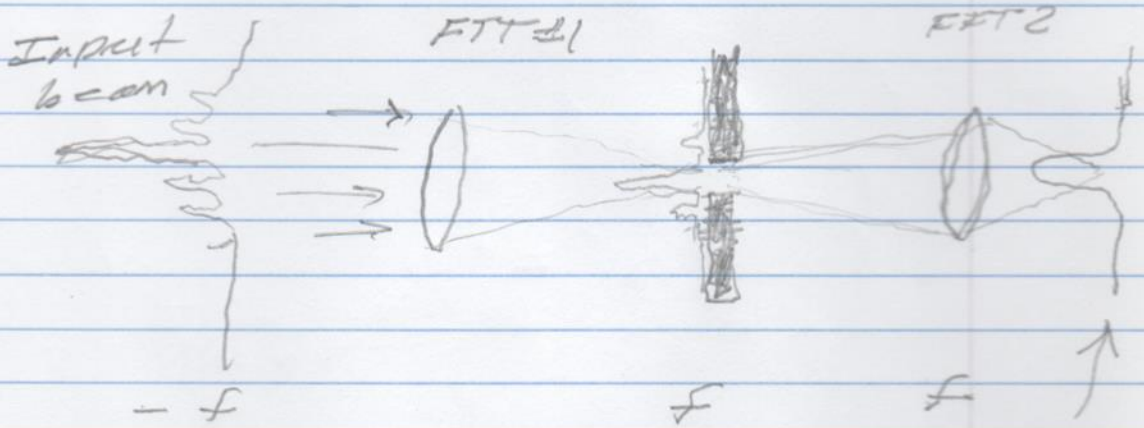
$$\therefore \mathcal{E}^f = (\lambda f)^2 \frac{\mathcal{E}_0\, \mathcal{E}\pi e^{ikr}}{2 i\lambda f} \frac{A_0^2}{A_0^2} e^{-\pi(x^2+y^2)/A_0^2}$$

Gaussian clean up with spatial filter continued.

i.e. $\mathcal{E}^f = \text{constant} * e^{-\pi(x^2+y^2)/A_0^2}$

where $A_0$ was orginal gaussian envelope

Picture of what we did:




Perfect Gaussian beam. Ideal for Holography or optical computing

$$FT\{f(x'y')\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(ux+vy)} e^{-(x^2+y^2)/A_0^2} \left\{1+\epsilon\cos\frac{2\pi x}{d}\right\} dx\,dy$$

but $FT\{f(x',y')\}(u,v) = FT\{gauss(x,y)\cdot mod(x)\}$

where $gauss(x,y) = e^{-(x^2+y^2)/A_0^2}$

$mod(x) = 1+\epsilon\cos\frac{2\pi x}{d}$

use inverse convolution theorem:

$FT[g(x)\cdot f(x)] =$ convolution of $FT(g)\, FT(h)$

convolution, $h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x')\, f(x-x')\, dx'$

so $$g(uv) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)A_0^2} e^{-2\pi i ux} e^{-2\pi i vy}\, dx\,dy$$

$$g = (\sqrt{\pi})^2 A_0^2\, e^{-\pi^2 A_0^2(u^2+v^2)}$$

$u = \frac{x}{\lambda f},\ v = \frac{y}{\lambda f}$

and $$f(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i x}\left(1+\epsilon\cos\frac{2\pi x}{d}\right)dx\, e^{-2\pi i}\, dy$$

$$= \delta(v)\cdot \underbrace{\int_{-\infty}^{\infty}\left(1+\epsilon\cos\frac{2\pi x}{d}\right)e^{-2\pi i x}\, dx}_{\delta u + \frac{1}{2}\left\{\delta(u+\frac{1}{d}) + \delta(u-\frac{1}{d})\right\}}$$

$$f(v,v) = \delta(v)\left\{\delta u + \tfrac{1}{2}\left(\delta(u+1/d) + \delta(u-1/d)\right)\right\}$$

Now :
$$\mathcal{FT}\left[g(x)f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x')f(x-x')dx'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{\pi}^{\,2} A_0^2 e^{-\pi^2 A_0^2(u'^2+v'^2)} \cdot du'dv'$$
$$\delta(v-v')\left\{\delta(u-u') + \frac{1}{2}\left(\delta(u+1/d-u') + \delta(u-\tfrac{1}{d}-u')\right)\right\}$$

Our spatial filter limits the frequency in the convolution
$A_0 = 5d$

$$\int_{-\infty}^{\infty} dv' \int_{-\infty}^{\infty} I(u')\,du' = \int_{-\infty}^{\infty} dv' \int_{-1/d}^{1/d} I(u')\,du'$$

Note $u = \frac{x}{\lambda f}$ $v = \frac{y}{-\lambda f_1}$

The sifting property of the delta function selects only the central diffracted spot

$$\therefore \quad \mathcal{FT} = \sqrt{\tfrac{\pi}{2}}\, A_0^2\, e^{-\pi^2 A_0^2(u^2+v^2)}$$

This is Intensity input to the last lens. Next, we solve for E-field after final lens.

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$$\mathcal{E}^f = \frac{\mathcal{E}_0 e^{ikr}}{i\lambda f} \; \mathcal{FT}\left[\sqrt{\tfrac{\pi}{2}}\, A_0^2\, e^{-\pi^2 A_0^2 (u^2+v^2)}\right]$$

Again $\mathcal{FT}$ gaussian = gaussian

$$\therefore \mathcal{E}^f = \frac{\mathcal{E}_0}{i\lambda f} A_0^2 \sqrt{\tfrac{\pi}{2}}\, e^{ikr} \iint_{-\infty}^{\infty} e^{-2\pi i u x} e^{-\pi^2 A_0^2 u^2} dx$$

$$\cdot\, e^{-2\pi i v y} e^{-\pi^2 A_0^2 v^2} dy$$

$u = \frac{x}{\lambda f} \qquad v = \frac{y}{\lambda f}$

$dv = \frac{1}{\lambda f} dy \qquad du = \frac{1}{\lambda f} dx$

$dy = \lambda f\, dv \qquad dx = \lambda f\, du$

$$\mathcal{E}^f = \frac{\mathcal{E}_0}{i\lambda f} A_0^2 \sqrt{\tfrac{\pi}{2}}\, e^{ikr} \int_{-\infty}^{\infty} e^{-2\pi i u x} e^{-\pi^2 A_0^2 u^2} \frac{du}{\lambda f}$$

$$\cdot \int_{-\infty}^{\infty} e^{-2\pi i v y} e^{-\pi^2 A_0^2 v^2} \frac{dv}{\lambda f}$$

$$\text{FT gaus } \frac{y}{w_0} = FT\left[e^{-u^2/w_0^2}\right] = \sqrt{\pi}\, w_0\, e^{-(\pi x w_0)^2}$$

$$\text{let } \frac{1}{w_0^2} = \pi^2 A_0^2$$

$$\mathcal{E}^f = \frac{\mathcal{E}_0}{i\lambda f} A_0^2 \sqrt{\tfrac{\pi}{2}} \frac{(\lambda f)^2}{(\lambda f)^2} e^{ikr} \cdot \int_{-\infty}^{\infty} e^{-u^2/w_0^2} e^{-2\pi i v y} dv$$

$I =$

$$\cdot \int_{-\infty}^{\infty} e^{-v^2/w_0^2} e^{-2\pi i u x} du$$

$$\mathcal{E}^f = -i\, \mathcal{E}_0 \frac{A_0^2}{\lambda f} \sqrt{\frac{\pi}{2}}\, e^{ik_2 r} \cdot \sqrt{\pi} w_0 \cdot \sqrt{\pi} w_0$$

$$\cdot\, e^{-\pi^2 w_0^2 x^2}\, e^{-\pi^2 w_0^2 y^2}$$

$$w_0^2 = \frac{1}{\pi^2 A_0^2}$$ eating up our $\pi^2$!

$$\Rightarrow A_0^2 = \frac{1}{\pi^2 w_0^2}$$

$$\mathcal{E}^f = -i\, \mathcal{E}_0 A_0^2 \cdot \pi w_0^2 \cdot \sqrt{\frac{\pi}{2}}\, e^{ik_2 r}$$

$$\cdot\, e^{-(x^2+y^2)/A_0^2}$$

Now, $A_0^2 \cdot \pi w_0^2 = \cancel{A_0^2}\, \pi \frac{1}{\pi^2 \cancel{A_0^2}} = \frac{1}{\pi}$

$$\mathcal{E}^f = -\frac{i\,\mathcal{E}_0}{\pi} \cdot \sqrt{\frac{\pi}{2}}\, e^{ik_2 r}\, e^{-(x^2+y^2)/A_0^2}$$

And so, Intensity $= \mathcal{E}^{f*} \cdot \mathcal{E}^f$

$$= \frac{i\,\mathcal{E}_0}{\sqrt{2\pi}}\, e^{-ik_2 r}\, e^{-(x^2+y^2)/A_0^2}$$

$$\cdot\, -i \frac{\mathcal{E}_0}{\sqrt{2\pi}}\, e^{ik_2 r}\, e^{-(x^2+y^2)/A_0^2}$$

$$= \frac{\mathcal{E}_0^2}{2\pi}\, e^{-2(x^2+y^2)/A_0^2}$$

Result is a clean gaussian beam.

Aside, some useful Fourier Transform bits:

1) May need use of a trig identity

a) $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ b) $e^{-i\theta} = \cos\theta - i\sin\theta$

$2\cos^2\theta = 1+\cos 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

2) spatial frequency

$u = \frac{x}{f\lambda}$ $\left[\frac{1}{\text{length}}\right]$ ; $v = \frac{y}{f\lambda}$

3) $\mathcal{FT}[g \circledast h]u = \mathcal{FT}(g)\mathcal{FT}(h)$

$\mathcal{FT}$ denotes Fourier Transform

Convolution: $\mathcal{FT}(f(x) \circledast g(x)) = \mathcal{FT}\int_{-\infty}^{\infty} f(x)g(u-x)du$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(u-x)e^{2\pi i k u}\,du$$

4) $\mathcal{FT}(1) = \int_{-\infty}^{\infty} 1\cdot e^{-2\pi i u x}\,dx = \delta(u)$

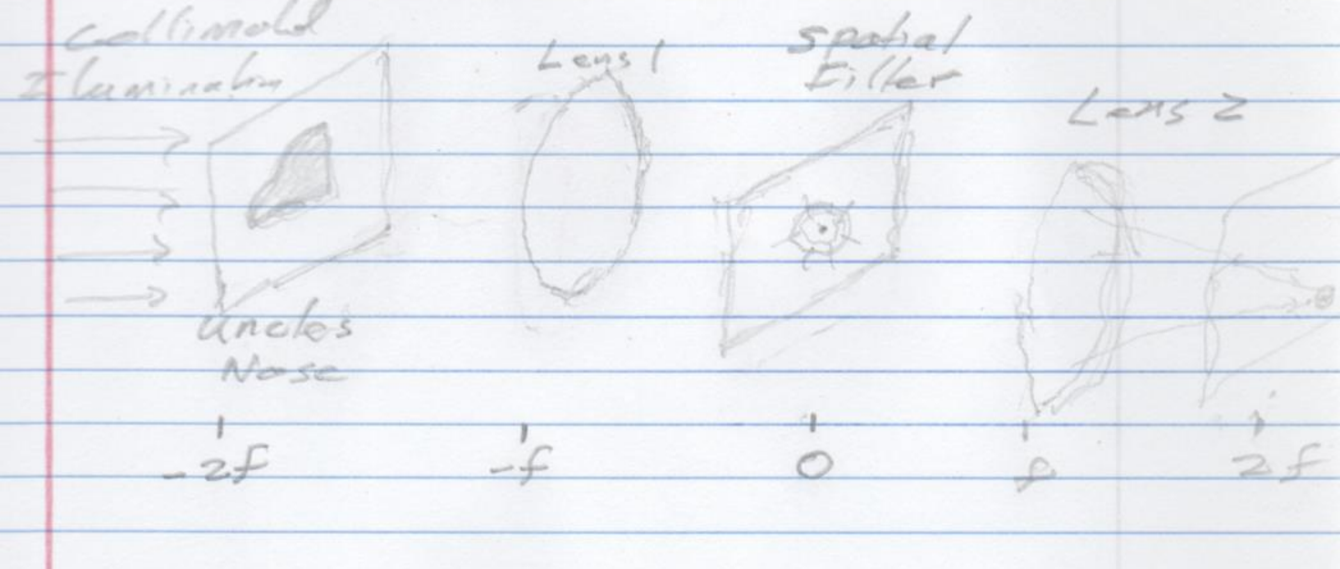
5) $\mathcal{FT}(\cos(2\pi u_0 x)) = \frac{1}{2}\{\delta(u+u_0) + \delta(u-u_0)$

6) $\mathcal{FT}[e^{-ax^2}](k) = \int_{-\infty}^{\infty} e^{-ax^2}e^{2\pi i k x}\,dx$

$$= \sqrt{\frac{\pi}{a}}\,e^{-\pi^2 k^2/a}$$

The optical aut corellator.

In optical corellation we use the "4f" spatial filter arrangement to make a yes/no decision on a part. Yes, the part looks like the one to be determined or no it does not resemble the object to be determined. Lets use an example of your uncles' nose.




A bright spot at 2f indicates that yes, we are looking at a transparency of your uncle's nose. We cant make a commercial product detecting your uncles nose, but, if we substitute a transparency of a virus, we can optically verify a virus in real time.