Chapter 2 Elements of Fourier or Physical Optics

Card Zeiss commissioned Abbe to make a better microscope. Abbe failed as he tried a small aperture system. He reported that diffraction was the issue. So he went to a larger diameter system and succeeded.

\[ \text{Intensity given by Zernike function} \]

\[ \text{Period} = \frac{\lambda}{2 \sin \delta} \]

\[ \tan \delta = \frac{D}{2f} = \frac{D}{2f} \]

Small angle \( \delta \approx \frac{D}{2f} \)

\[ \sin \delta \approx \frac{D}{2f} \]

Period or spatial frequency

\[ = \frac{\lambda}{D/2f} - \frac{\lambda f}{D} \]

Similar to Frenkel's result.
Elements of Fourier optics or Physical optics p. 2

Fourier optics, light field is written as a superposition of plane waves propagating in different directions. A distribution of angles is akin to "angular frequency."

The wavelets, Huygens principle are comprised of a superposition of angles. The result of len depends upon the spatial frequency.

Spatial frequency

\[ u = \frac{x}{f\lambda} \]  \[ v = \frac{x}{f\lambda} \]

Since \[ f = \text{[length]} \] \[ \lambda = \text{[length]} \]

\[ u = \text{[1/length]} \]
Elements of Fourier or Physical optics. P. 3

Fresnel distortion integral

\[ E^f = \left( \frac{\varepsilon_0 \varepsilon \delta^*}{(i \Delta f)^2} \right) \mathcal{F} \left[ \frac{f(x', y')^2}{2f} \right] \]

Fresnel propagator of space \( x' \)

\[ r = f + \frac{p^2}{2f} \]

For ease of solution, we split the Fresnel integral into two parts.

1) \( f(x, y) \) = field incident on lens.
2) \( t(x, y) \) = transmission of lens.

With the convolution theorem we have:

\[ E^f = \frac{\varepsilon_0 \varepsilon \delta^*}{i \Delta f} \mathcal{F} \left[ f(x', y')^{2f} \right] \mathcal{F} \left[ t(x', y') \right] \]

Point spread function

\[ = \frac{\varepsilon_0 \varepsilon \delta^*}{i \Delta f} \mathcal{F} \left[ f(x', y') \right] \text{ PSF} \]

For a circular lens

point spread, \( p(x, y) = \frac{\pi D^2}{4} \text{jinc} \left( \frac{\pi D}{2f} \right) \]

\text{jinc} is Bessel function analog of \( \text{sinc} = \frac{\sin x}{x} \)

Adams & Hughes "Fourier optics" p. 1.

Fresnel to Fourier Oxford 2019

Spence, High resolution section 3.5
Elements of Fourier Optics

For a plane wave on a long thin lens, the solution is simple:

\[ f(x',y') = e^{i\phi} \]

\[ \mathcal{F}[e^{i\phi}] = \delta(u) \cdot \text{delta function} \]

For two plane waves separated by an angle \( \Delta \theta \) we have:

\[ \mathcal{F}\left[ e^{i(u'\cos \theta - v'\sin \theta)} \right](u) = \delta(u - \frac{\Delta \theta}{\lambda}) \]

where \( \frac{2\pi (u - \Delta \theta)}{\lambda} = \Delta \theta \cdot \frac{u}{\lambda} \).

Plugging this into Fresnel's eq.

\[ \mathcal{F} = \int \frac{e^{i2\pi xu/\lambda} \delta(u - \Delta \theta)}{\lambda} \cdot \text{p.s.f.} \]

For a lens of diameter \( D \) and focal length \( f \):

\[ \text{p.s.f.} = \text{jinc}\left(\frac{\pi D \theta}{\lambda f}\right) \]

Intensity \( \mathcal{I} = \mathcal{E}^* \mathcal{E} \)

\[ \mathcal{I} = \frac{T_0 \pi^2 D^4}{16\pi^2 f^2} \text{jinc}^2\left\{ \frac{\pi D(u - \Delta \theta)^2 + v^2}{\lambda f} \right\}^{1/2} \]

Air Pattern center displaced to \( x = \Delta \theta f \)

see eq 9.5 Adams + Hughes
Elements of Fourier optics

So, the two monochromatic plane waves of slightly different incident angle, Δθ, give us a nice result. The observed intensity in the focal plane is two Airy patterns. What is the resolution of the system given these two incident plane waves?

Rayleigh criteria:

Airy = 1

\[ f Δθ = \frac{1.22 \lambda}{D} \]  (left as homework)

\[ Δθ_r = \frac{1.22 \lambda}{D} \]

so we get smaller Δθ, better resolution when D is increased. That is, when more spatial frequencies are subtended by the lens.
Spatial filtering.

We learned that resolution increases when we increase the diameter of the lens; this confounded Abbe early on.

Can we alter the resolution of a system by changing the spatial frequencies that get to the lens? Yes, concept of apodization, to cut feet off a pod = without feet.

We do this in optical holography, "clean up" the laser beam with a strong lens 1, objective eyepiece, and a pin hole aperture at focal point.
Fourier optics

We see examples of the Fourier transform of the lens in Appendix II. Physical optics

Appendix II. Fig. 1 sinc (lambdas)
rectangular aperture

Fig. 2 Bessel functions
role different orders
and symmetry

Fig. 3 Zinic function,
$FT$ of round lens

From the result of Fig. 4, we see the Zinic and sinc function are very similar. So, if you are familiar with sinc$(x) = \frac{\sin(x)}{x}$ please use that analogy.

Airy function, the intensity or
field squared at focal point

$E^2 = \frac{E_0^2}{iaf} f \left[ f(x', y') \right](uv)*p5f*(uv)$

for plane wave $f(x, y) = \frac{1}{i\lambda x}$

$p5f(u, v) = \frac{-iD^2}{\lambda^2} \text{zinc}\left(\frac{\pi D}{\lambda^2}\right)$

where $u = \frac{x}{f}$, $v = \frac{y}{f}$
Fourier analysis p. 8

But Fourier transform of a plane wave = $\delta$ function

\[ \mathcal{F}(\delta) = \frac{\delta}{i\omega} \]

For a plane wave inclined at an angle $\alpha$ to normal incidence, this example used in Adams & Hughes as an example of angular resolution.

\[ I(\alpha) = \frac{I_0 \pi^2 D^4 \sin^2 \alpha \left( \frac{\omega - \omega_0}{\omega_0} \right)^4}{16\lambda^2 z^2} \]

See appendix II fig 4.

Rign's pattern + fig 5 have overlapping dark patterns.

The angular resolution of a lens is the separate minimum separation of the air patterns, or when $\max$ of one pattern $\Delta \theta$ is equal to $\min$ of other

\[ \Delta \theta_{\text{resolution}} = \frac{1.22 \lambda}{D} \]

This is what Röntgen discovered. We need a bigger lens to improve the microscope.
Inverse Abelization, super resolve.

You can actually block the central part of a diffraction pattern (image) of lens 1 with an annulled filter, then re-image this with lens 2 to form an image with only the higher spatial frequencies and "increase" the resolution.

Conversely, in dark field TEM we select only a few spatial frequencies. This eliminates the higher spatial frequencies and so the image has lower resolution.
Fourier Optics page 10

Relationships between Fourier optics concepts and terminology and electron microscopy.

Does the point spread function of page 9, \( f(x,y) \), which is typically \( \pi \alpha^2 \cdot \text{jinc} (\pi \alpha x) \) for Adams, equal to the contrast transfer function in electron microscopy of space or contrast williams?

Yes, space, has similar approach but the psf, point spread function, of Fourier optics is modulated by a term that represents the strong influence of spherical aberration in electron optics.

\[
A(u,v) = P(x) \exp(jx(u + v))
\]

In Williams and Carter, intensity g(x,y):

\[
g(x,y) = \sum G(u,v) \exp 2\pi j(xu + vy)
\]

In the case where we are reciprocal lattice vec.
Fourier optics

Relationship of FP Fourier optics to modern electron optics terms:

In an electron microscope, the electron beam (charged particle) interacts with the charge in the sample. This changes the phase of the wave electron waves. This is a more complicated interaction than the non-charged photon (light) passing through a sample at a lens.

The connection of Fourier optics to electron microscopy is particularly strong in the TEM. In the back focal plane of the objective lens we have the diffraction pattern or the angular information:

Beam is scattered by sample

Objective lens

Back focal plane

Scattered beams from diffraction collected in SPD

Now, back to Fourier optics
Example, if spatial filter.

Or, cleaning up a Gaussian laser beam. On the bottom of page 4, I mentioned cleaning up a laser beam with a strong lens + pin hole aperture. Now we see the math of this.

\[ f(x', y') = e^{-\frac{(x'^2 + y'^2)}{A_0^2}} \left( 1 + e^{\cos \left( \frac{2\pi x'}{d} \right)} \right) \]

where \( A_0 \) = Gaussian beam waist.

\( d \) = spacing between the diode and lens

\[ \text{Fresnel integral:} \]

\[ \varepsilon(t) = \frac{E_0 e^{-4\pi r}}{\pi ft} \int_{-\infty}^{\infty} e^{-\left( \frac{\left( (x'^2 + y'^2) / A_0^2 \right)}{(1 + e^{\cos (2\pi x'/d)})} \right)} dx' dy \]

\[ \text{FT}\{f(x, y') \mu \nu \} = \text{FT}\left\{ e^{-\left( \frac{(x'^2 + y'^2)}{A_0^2} \right)} \left( 1 + e^{\cos \left( \frac{2\pi x'}{d} \right)} \right) \right\} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i (ux + vy)} e^{-\left( \frac{(x'^2 + y'^2)}{A_0^2} \right)} \left( 1 + e^{\cos (2\pi x'/d)} \right) dx' dy \]

This is hard to integrate, so let's examine the convolution approach.
Clearing a gaussian beam continued:

Now carry out the convolution:

\[ E_f = \frac{\mathcal{E}_0 e^{i k r}}{i k f} \left\{ \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right\} \left( \frac{1}{\pi A_0^2} e^{-\pi (x^2 + y^2 + z^2) A_0^2} \right) \]

\[ A_0 = \text{width of Gaussian beam} \]

Take only central spatial freq.

Set \[ \frac{A_0}{\lambda} = 5 \Rightarrow \frac{1}{f} = 5 A_0 \]

This eliminates the two spots from the spherical functions.

\[ E_f = \left( \frac{\mathcal{E}_0 e^{i k r}}{2 i k f} \right) \left( \frac{1}{\pi A_0^2} e^{-\pi (x^2 + y^2 + z^2) A_0^2} \right) \left( \delta(x) \right) \]

Now Fourier Transform again at the first focal plane.

\[ -2f \quad -f \quad 0 \quad f \quad 2f \]
Fourier optics

If spatial filter combined, Gaussian beam clean up:

After the second lens, one more FT.

\[ z = \frac{z_0}{f} e^{-\frac{2\pi}{\lambda} (x^2 + y^2) - \frac{2\pi}{\lambda} (x y)} \]

integral over \( dv \) as usual?

See if fill example.

Note Sec 9.6 Adams & Mather,

\[ u = \frac{x}{f\lambda}, \quad v = \frac{y}{f\lambda} \]

Now, use \( e^{iu} = \cos u - i\sin u \)

Integral over sine variables

Then Abramowitz Stegun, 10.12 p. 302

\[ \mathcal{F}\left[e^{-\alpha x^2}\right] = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{2\pi}{\lambda} x^2} \frac{1}{\pi^{3/2}} \]

\[ \Rightarrow \mathcal{F}\left[e^{-\pi x^2/A_0^2}\right] = \sqrt{\frac{\pi}{\pi A_0^2}} e^{-\frac{\pi}{2} x^2/A_0^2} \]

\[ = \frac{1}{A_0} e^{-\frac{\pi}{2} x^2/A_0^2} \]

\( A_0 = \text{width of Gaussian,} \)

so \( x^2/A_0^2 \) has no dimensions.

\[ z = \frac{(df)^2}{2i\lambda f^2} \text{ with } A_0^2 e^{-\frac{\pi}{2} x^2/A_0^2} \]
Gaussian clean up with spatial filter continued.

\[ \phi = \text{constant} \times e^{-\pi (x^2 + y^2)/w_0^2} \]

where \( w_0 \) was original gaussian envelope.

Picture of what we did:

Input beam \[ \rightarrow \text{FTT 1} \rightarrow \text{FTT 2} \rightarrow \text{Perfect Gaussian beam} \, \text{Ideal for holography or optical computing} \]
Fourier Optics

\[ \mathcal{F} \{ \mathbf{f}(x, y) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i (ux + vy)} e^{-(x^2 + y^2)/a_0^2} \left( 1 + e^{-2\pi i x} \right) dx dy \]

but \( \mathcal{F} \{ \mathbf{f}(x, y) \} = \mathcal{F} \{ \text{Gauss}(x, y) \} \cdot \text{mod}(x) \)

where \( \text{Gauss}(x, y) = e^{-\left(x^2 + y^2\right)/a_0^2} \)

\[ \text{mod}(x) = 1 + e^{-2\pi i x} / 2 \]

Use inverse convolution theorem:

\[ \mathcal{F}^{-1} \left[ \mathcal{F} \{ g(x) \} \cdot \mathcal{F} \{ h(x) \} \right] = \text{convolution of } \mathcal{F}^{-1} \{ g(x) \} \cdot \mathcal{F}^{-1} \{ h(x) \} \]

\[ \text{Convolution, } h(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} g(x) \cdot f(x-x') \, dx' \]

So \( g(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2 + y^2\right)/a_0^2} e^{-2\pi i ux} e^{-2\pi i vy} \, dx \, dy \)

\[ y = (\sqrt{\pi})^2 a_0 \cdot e \]

\[ \mathcal{F} \{ \text{Gauss}(x, y) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2 + y^2\right)/a_0^2} e^{-2\pi i xy} \, dx \, dy \]

and \( f(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i ux} (1 + e^{-2\pi i x}) \, dx \, dy \)

\[ f(u, v) = \delta(v) + \int_{-\infty}^{\infty} \left( 1 + e^{-2\pi i x} \right) e^{-2\pi i x} \, dx \]

\[ f(u, v) = \delta(v) \left( 5u + 1 \left( 5u^2 + 1 \right) \right) \]

So \( f(u, v) = \delta(v) \left\{ 5u + 1 \left( 5u^2 + 1 \right) \right\} \)
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Now:

\[ \mathcal{F}^T \left[ g(x) f(x) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x') f(x-x') dx' \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\pi A_0^2 e^{-\pi^2 A_0^2 (u'^2 + v'^2)} \delta(u-v) \delta(u'-v') \left\{ \delta(u-u') + \frac{1}{2} \left( \delta(u+v-u') + \delta(u-v-u') \right) \right\} \]

Our spatial filter limits the frequency in the convolution.

\[ A_0 = 5d \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(u') du' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(u) du \]

Note \( u = \frac{x}{x'^2} \), \( v = \frac{y}{y'^2} \).

The sifting property of the delta function selects only the central diffracted spot.

\[ -\pi^2 A_0^2 (u^2 + v^2) \]

\[ \frac{\partial^2}{\partial T} = -\sqrt{2} \pi A_0^2 e \]

This is the last input to the last lens. Next, we solve for 

The final lens.
\[ E = \frac{E_0 e^{i\theta}}{i\lambda} \int \left[ \sqrt{\pi} A_0^2 e^{-\frac{2\pi^2 A_0^2 x^2}{\lambda^2}} \right] dx \]

Again, \( FT \) Gaussian = Gaussian.

\[ \therefore E = \frac{E_0 A_0^2 \sqrt{\pi}}{i\lambda} \int \frac{e^{i\theta} e^{-2\pi i u x - \pi^2 A_0^2 u^2}}{e^{-2\pi i v y - \pi^2 A_0^2 v^2}} \frac{du}{\lambda} \frac{dv}{\pi} \]

\[ u = \frac{x}{\lambda}, \quad v = \frac{y}{\lambda} \]

\[ dv = \frac{1}{\lambda} dy, \quad du = \frac{1}{\lambda^2} dx \]

\[ dy = \frac{2\pi}{\lambda} dv, \quad dx = i\lambda du \]

\[ E = \frac{E_0 A_0^2 \sqrt{\pi}}{i\lambda} \int \left[ e^{i\theta} e^{-2\pi i u x - \pi^2 A_0^2 u^2} \right] \left[ e^{-2\pi i v y - \pi^2 A_0^2 v^2} \right] \frac{du}{\lambda} \frac{dv}{\pi} \]

\[ FT \) Gaussian = \int \left[ e^{-u^2/4w_0^2} \right] = \sqrt{\pi w_0} e^{-\frac{1}{4w_0^2}} \]

\[ \frac{1}{\lambda} + \frac{1}{w_0} = \pi^2 A_0^2 \]

\[ E = \frac{E_0 A_0^2 \sqrt{\pi}}{i\lambda} \int \frac{e^{i\theta} e^{-u^2/4w_0^2 - 2\pi i u x - \pi^2 A_0^2 u^2}}{e^{-2\pi i v y - \pi^2 A_0^2 v^2}} \frac{du}{\lambda} \frac{dv}{\pi} \]

\[ I = \sqrt{\pi w_0} e^{-\frac{1}{4w_0^2}} \]
\[ \Sigma_f = -i \frac{\varepsilon_0 A_0^2}{\sqrt{2\pi}} e^{-i \xi r} \sqrt{\frac{\pi}{\xi}} \frac{1}{\sqrt{2 \xi w_0^2}} \frac{1}{\sqrt{2 \xi w_0^2}} \]

\[ w_0^2 = \frac{1}{\eta^2 A_0^2} \text{ canceling up our } \pi^2! \]

\[ \Rightarrow A_0^2 = \frac{1}{\eta^2 w_0^2} \]

\[ \Sigma_f = -i \frac{\varepsilon_0 A_0^2}{\sqrt{2\pi}} \pi w_0^2 = \sqrt{\pi} e^{-i \xi r} \]

\[ e^{-\frac{(x^2+y^2)}{A_0^2}} \]

Now, \[ A_0^2 \cdot \pi w_0^2 = A_0^2 \frac{\pi}{\pi^2 A_0^2} = \frac{1}{\eta^2} \]

\[ \Sigma_f = -i \frac{\varepsilon_0}{\sqrt{2\pi}} \sqrt{\pi} e^{-i \xi r} e^{-\frac{(x^2+y^2)}{A_0^2}} \]

And so, intensity \[ = \Sigma_f \ast \Sigma_f \]

\[ = -i \frac{\varepsilon_0}{\sqrt{2\pi}} e^{-i \xi r} e^{-\frac{(x^2+y^2)}{A_0^2}} \]

\[ = -i \frac{\varepsilon_0}{\sqrt{2\pi}} e^{-i \xi r} e^{-\frac{(x^2+y^2)}{A_0^2}} \]

\[ = \frac{\varepsilon_0^2}{2\pi} e^{-2 \frac{(x^2+y^2)}{A_0^2}} \]

Result is a clean gaussian beam.
Aside, some useful Fourier Transform bits:

1) Many need use of a trig identity
   a) \( \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \)
   b) \( e^{i \theta} = \cos \theta + i \sin \theta \)

   \[ 2 \cos^2 \theta = 1 + \cos 2\theta \]
   \[ \cos 2\theta = 2 \cos^2 \theta - 1 \]

2) Spatial frequency
   \[ \mu = \frac{x}{f' \lambda} \] \[ \nu = \frac{y}{f' \lambda} \]

3) \( \mathcal{F} \mathcal{T}[g \cdot h] = \mathcal{F} \mathcal{T}[g] \cdot \mathcal{F} \mathcal{T}[h] \)
   \( \mathcal{F} \mathcal{T} \) denotes Fourier Transform

   Convolution: \( \mathcal{F} \mathcal{T} [f(x) \cdot g(x)] = \mathcal{F} \mathcal{T} [f(x) \cdot g(-x)] \cdot \mathcal{F} \mathcal{T} [g(x) \cdot \delta(x)] \)
   \[ = \int \int \mathcal{F} \mathcal{T} [f(x) \cdot g(-x)] \cdot \mathcal{F} \mathcal{T} [g(x) \cdot \delta(x)] \cdot dx \]

4) \( \mathcal{F} \mathcal{T}(1) = \int_{-\infty}^{\infty} 1 \cdot e^{-2\pi i x} \cdot dx = \delta(x) \)

5) \( \mathcal{F} \mathcal{T} [\cos(2\pi \nu x)] = \frac{1}{2} \left( \delta(\nu + \frac{m}{\lambda}) + \delta(\nu - \frac{m}{\lambda}) \right) \)

6) \( \mathcal{F} \mathcal{T} [e^{-ax^2}](k) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-ax^2} \cdot e^{2\pi i kx} \cdot dx \)
   \[ = \frac{1}{\sqrt{a}} \cdot e^{-\pi^2 k^2 / a} \]
The optical art correlator.

In optical correlator, we use the 4f spatial filter arrangement to make a yes/no decision on a part. Yes, the part looks like the one to be determined or no it does not resemble the object to be determined. Let's use an example of your uncle nose.

A bright spot at 2f indicates that yes, we are looking at a transparency of your uncle's nose. We can't make a commercial product detecting your uncle's nose, but if we substitute a transparency of a virus, we can optically verify a virus in real time.